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Stability of the trivial solution for linear stochastic differential equations with Poisson white noise

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Abstract
Two methods are considered for assessing the asymptotic stability of the trivial solution of linear stochastic differential equations driven by Poisson white noise, interpreted as the formal derivative of a compound Poisson process. The first method attempts to extend a result for diffusion processes satisfying linear stochastic differential equations to the case of linear equations with Poisson white noise. The developments for the method are based on Itô's formula for semimartingales and Lyapunov exponents. The second method is based on a geometric ergodic theorem for Markov chains providing a criterion for the asymptotic stability of the solution of linear stochastic differential equations with Poisson white noise. Two examples are presented to illustrate the use and evaluate the potential of the two methods. The examples demonstrate limitations of the first method and the generality of the second method.

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1. Introduction

Formation of dynamic patterns in fluids, transition from uncorrelated to coherent oscillations in lasers, crystal growth, flatter of airplane wings, population dynamics and other phenomena in physics, engineering and biology can be described by evolution equations with noise [4, 5]. Generally, these evolution equations are nonlinear and stochastic with continuous or discrete time. The continuous and discrete time evolution equations, referred to as stochastic differential equations and discrete noise maps, are driven by Gaussian white noise processes, defined as the formal derivative of Brownian motions, and sequences of independent identically distributed random variables, respectively ([4], chapters 4 and 11).

We consider stochastic differential equations driven by Poisson white noise, defined by the formal derivative of a compound Poisson process. The Poisson white noise can be viewed as a sequence of independent identically distributed random pulses arriving at random times given by the jump times of a Poisson process. There are at least three reasons for studying evolution equations with Poisson white noise. First, the contribution of microscopic processes to evolution equations established at a mezoscopic level can be captured by adding noise sources ([4], chapter 10). Poisson white noise is a natural model for some noise sources, for example, sources corresponding to impacts between microscopic particles. Second, the Gaussian white noise can be viewed as the limit of a sequence of Poisson white noise processes with pulses of increasing frequency and decreasing magnitude. Third, methods developed for the analysis of Markov chains can be used to study the long term behaviour of the solution of stochastic differential equations with Poisson white noise since these equations can be recast as discrete noise maps, as demonstrated in the paper.

Let $\tilde{\boldsymbol{X}}$ be an $\mathbb{R}^d$-valued stochastic process defined by the evolution equation

$$\dot{\tilde{\boldsymbol{X}}}(t) = \boldsymbol{g}(\tilde{\boldsymbol{X}}(t), \boldsymbol{\alpha} + \boldsymbol{W}(t)), \qquad t \geqslant 0, \qquad \tilde{\boldsymbol{X}}(0) = \boldsymbol{x}_0 \in \mathbb{R}^d, \tag{1}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^{d'}$ is a parameter, $\boldsymbol{W} \in \mathbb{R}^{d'}$ denotes a white noise process and $d, d' \geqslant 1$ are integers. Suppose that equation (1) has a stationary solution $\tilde{\boldsymbol{X}}_s$. A main objective of stochastic stability studies is the determination of subsets in $\mathbb{R}^{d'}$ consisting of values of $\boldsymbol{\alpha}$ for which solutions $\tilde{\boldsymbol{X}}$ of equation (1) starting in a small vicinity of a stationary solution $\tilde{\boldsymbol{X}}_s$ converge to $\tilde{\boldsymbol{X}}_s$ as time increases indefinitely. If this behaviour is observed for almost all samples of $\tilde{\boldsymbol{X}}$, then $\tilde{\boldsymbol{X}}_s$ is said to be stable a.s.

We determine whether a stationary solution $\tilde{\boldsymbol{X}}_s$ of equation (1) is or is not a.s. stable by studying the evolution in time of the difference $\boldsymbol{X} = \tilde{\boldsymbol{X}} - \tilde{\boldsymbol{X}}_s$, where $\tilde{\boldsymbol{X}}$ is a solution of equation (1) representing a perturbation about $\tilde{\boldsymbol{X}}_s$. The stationary solution $\tilde{\boldsymbol{X}}_s$ is a.s. stable if the trivial stationary solution of the defining equation for $\boldsymbol{X}$ is stable almost surely. Since $\|\boldsymbol{X}(t)\|$ is small at least initially, $\boldsymbol{X}$ satisfies approximately a differential equation obtained from equation (1) by linearization about $\tilde{\boldsymbol{X}}_s$, that is, $\boldsymbol{X}$ is the solution of a linear stochastic differential equation.

If $\boldsymbol{W}$ in equation (1) is a Gaussian white noise, then $\boldsymbol{X}$ is an $\mathbb{R}^d$-valued diffusion process satisfying a stochastic differential equation whose drift and diffusion coefficients are linear functions of the state $\boldsymbol{X}$. The Gaussian white noise is viewed as the formal derivative of a Brownian motion process. Conditions have been established in [6] under which the diffusion process $\boldsymbol{X}$ is such that $\|\boldsymbol{X}(t)\|$ converges a.s. to 0 as $t \to \infty$, that is, $P(\lim_{t\to\infty} \|\boldsymbol{X}(t)\| = 0) = 1$, irrespective of the initial state $\boldsymbol{X}(0) = \boldsymbol{x} \in \mathbb{R}^d$. If the above limit holds, it is said that the trivial solution of the defining stochastic differential equation for $\boldsymbol{X}$ is asymptotically stable almost surely. It was also shown in [6] that the top Lyapunov exponent can be used to assess whether the trivial solution is or is not stable.

As previously stated, we assume that $\boldsymbol{W}$ in equation (1) is a Poisson rather than Gaussian white noise, so that $\boldsymbol{X}$ is the solution of a linear stochastic differential equation driven by Poisson white noise. Two methods are considered for assessing the asymptotic stability of the trivial solution for this type of equations. The first method is an attempt of extending a result in [6], and uses Lyapunov exponents to forecast the long term behaviour of $\boldsymbol{X}$. Derivations are based on Itô's formula for semimartingales. The second method uses a geometric ergodic theorem for Markov chains to assess the long term behaviour of a Markov chain $\boldsymbol{X}_n$ associated with $\boldsymbol{X}$, and involves two steps. First, it is established whether the Markov chain $\boldsymbol{X}_n$ is or is not ergodic. Second, if $\boldsymbol{X}_n$ is ergodic, its long term behaviour can be inferred from the defining recurrence relationship for the Markov chain $\boldsymbol{X}_n$ or approximations of the invariant measure of this chain. The examples in the paper (1) demonstrate difficulties related to the extension of the method in [6] to the case of stochastic differential equations with Poisson white noise and (2) show that the second method provides a general criterion for assessing the long term behaviour of $\boldsymbol{X}_n$ and $\boldsymbol{X}$.

2. Problem definition

Let $\boldsymbol{X}$ be an $\mathbb{R}^d$-valued process defined on a probability space $(\Omega, \mathcal{F}, P)$ by the stochastic differential equation

$$\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{a}\boldsymbol{X}(t-)\,\mathrm{d}t + \left(\sum_{k=1}^{d} \boldsymbol{b}^{(k)} X_k(t-)\right) \mathrm{d}\boldsymbol{C}(t), \qquad t \geqslant 0, \tag{2}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}^{(k)}, k = 1, \ldots, d$, are (d, d) and (d, d') matrices with constant entries. The input $\boldsymbol{C}$ is an $\mathbb{R}^{d'}$-valued stochastic process with independent coordinates

$$C_r(t) = \sum_{q=1}^{N_r(t)} Y_{r,q}, \qquad r = 1, \ldots, d', \tag{3}$$

where N_r are Poisson processes with intensities $\lambda_r, r = 1, \ldots, d'$, that are independent of each other, $Y_{r,q}, q = 1, 2, \ldots,$ denote independent copies of $Y_{r,1}$ and $Y_{r,1}, r = 1, \ldots, d'$, are mutually independent real-valued random variables, which do not depend on the Poisson processes N_r.

Since the coefficients in equation (2) are linear functions of $\boldsymbol{X}$, this equation has a unique solution, which is a semimartingale ([8], theorem 6, p 194 and theorem 7, p 197). Our objective is to find conditions under which $\boldsymbol{X}$ converges almost surely to the origin of $\mathbb{R}^d$ as time increases indefinitely, that is, assess the stochastic stability for the trivial solution for equation (2). As previously stated, two methods are considered. The first is based on Lyapunov exponents, and the second uses concepts of stochastic stability for Markov chains.

As previously stated the Poisson white noise can be interpreted as the formal derivative of compound Poisson processes of the type in equation (3). The Poisson white noise consists of pulses of amplitude $Y_{r,q}$ arriving at the jump times of N_r, so that the average time between its consecutive pulses is $1/\lambda_r$. The Poisson white noise provides a realistic model for impacts between microscopic particles, such as atoms and molecules, and its limit for jumps of decreasing amplitude and increasing arrival rate approaches the Gaussian white noise.

3. Associated Markov chain

Let $\boldsymbol{X}$ be the process in equation (2) and consider a Poisson process N with intensity $\lambda = \sum_{r=1}^{d'} \lambda_r$. Set $T_0 = 0$ and denote by $0 < T_1 < T_2 < \cdots$ the jump times of N. The compound Poisson process $\boldsymbol{C}$ in equation (2) can be given in the form

$$\boldsymbol{C}(t) = \sum_{k=1}^{N(t)} \boldsymbol{V}_k, \tag{4}$$

where $\boldsymbol{V}_k$ are independent copies of an $\mathbb{R}^{d'}$-valued random variable with coordinates $V_{k,r} =$ an independent copy of $Y_{r,1}$ and $V_{k,p} = 0,\ p \neq r$, with probability $\lambda_r/\lambda, r = 1, \ldots, d'$.

Consider the sequence $\boldsymbol{X}_n = \boldsymbol{X}(T_n), n = 0, 1, \ldots,$ consisting of the values of $\boldsymbol{X}$ immediately following the jumps of N. The process $\boldsymbol{X}$ satisfies the deterministic differential equation $\mathrm{d}\boldsymbol{X}(t) = \boldsymbol{a}\boldsymbol{X}(t)\,\mathrm{d}t$ in the time interval $[T_{n-1}, T_n)$ so that $\boldsymbol{X}(T_n-)$ and $\boldsymbol{X}_{n-1}$ are related by $\boldsymbol{X}(T_n-) = \exp(\boldsymbol{a}Z_n)\boldsymbol{X}_{n-1}$, where $Z_n = T_n - T_{n-1}$ and $\exp(\boldsymbol{a}Z_n)$ denotes a matrix exponential ([1], section 1.5). The relationship between $\boldsymbol{X}(T_n-)$ and $\boldsymbol{X}_n$ is (equation (2))

$$\boldsymbol{X}_n = \boldsymbol{X}(T_n-) + \left(\sum_{k=1}^{d} \boldsymbol{b}^{(k)} X_k(T_n-)\right) \boldsymbol{Y}_n, \tag{5}$$

so that

$$\boldsymbol{X}_n = \mathrm{e}^{\boldsymbol{a}Z_n}\boldsymbol{X}_{n-1} + \left(\sum_{k=1}^{d} \boldsymbol{b}^{(k)} (\mathrm{e}^{\boldsymbol{a}Z_n}\boldsymbol{X}_{n-1})_k\right) \boldsymbol{Y}_n, \qquad n \geqslant 1. \tag{6}$$

The above formula shows that the sequence $\boldsymbol{X}_n, n = 0, 1, \ldots,$ is a Markov chain. Since the coefficients $\boldsymbol{a}$ and $\boldsymbol{b}$ in equation (2) do not depend explicitly on time, the one-step transition function $P(\boldsymbol{\xi}, A) = P(\boldsymbol{X}_n \in A \mid \boldsymbol{X}_{n-1} = \boldsymbol{\xi})$, $A \in \mathcal{B}(\mathbb{R}^d)$, $\boldsymbol{\xi} \in \mathbb{R}^d$, of the Markov chain $\boldsymbol{X}_n$ does not depend on n, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-field on $\mathbb{R}^d$. The m-step transition functions of $\boldsymbol{X}_n$ can be calculated for any integer $m \geqslant 1$ from the recurrence formula

$$P^{(m+1)}(\boldsymbol{\xi}, A) = \int_{\mathbb{R}^d} P^{(m)}(\boldsymbol{\eta}, A) P(\boldsymbol{\xi}, \mathrm{d}\boldsymbol{\eta}), \qquad m \geqslant 1, \tag{7}$$

with $P^{(1)}(\cdot,\cdot) = P(\cdot,\cdot)$. We also note that, for a fixed $\boldsymbol{\xi} \in \mathbb{R}^d$ and integer $m \geqslant 1$, $P^{(m)}(\boldsymbol{\xi}, \cdot)$ is a probability measure on $\mathcal{B}(\mathbb{R}^d)$.

4. An ergodic theorem for Markov chains

We state without proof a geometric ergodic theorem providing sufficient conditions for a Markov chain to be ergodic ([7], theorem 15.0.1). The statement of the theorem involves some less familiar concepts, which are discussed prior to its statement.

A Markov chain $\boldsymbol{X}_n, n = 0, 1, \ldots,$ is said to be ψ-irreducible if ψ is a measure on the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$ such that whenever $\psi(A) > 0$, $A \in \mathcal{B}(\mathbb{R}^d)$, the probability that the chain ever enters A is strictly positive for all starting points $\boldsymbol{X}_0 = \boldsymbol{\xi} \in \mathbb{R}^d$. A Borel set C in $\mathbb{R}^d$ is said to be small if there exists an integer $m > 0$ and a non-trivial measure ν_m on $\mathcal{B}(\mathbb{R}^d)$, referred to as minorizing measure, such that $P^{(m)}(\boldsymbol{\xi}, A) \geqslant \nu_m(A)$ for all $\boldsymbol{\xi} \in C$ and Borel sets A in $\mathbb{R}^d$, where $P^{(m)}$ is given by equation (7). A related, and somewhat weaker notion, is that of a petite set ([7], section 5.5.2). Let $\boldsymbol{X}_n, n = 0, 1, \ldots,$ be a irreducible Markov chain and let C be a ν_m-small set in $\mathbb{R}^d$ which can be taken to satisfy $\nu_m(C) > 0$. Hence, $P^{(m)}(\boldsymbol{\xi}, \cdot) \geqslant \nu_m(\cdot)$, $\boldsymbol{\xi} \in C$, so that the chain returns to C in m transitions with positive probability. Let $\mathcal{T}_C$ denote the time points for which C is a small set with minorizing measure proportional to ν_m. The greatest common denominator of $\mathcal{T}_C$ is the period for the set C. A Markov chain is aperiodic if the greatest common denominator of $\mathcal{T}_C$ is 1. This property does not depend on the choice of a set C as above.

And now we state the part of the geometric ergodic theorem for Markov chains in [7], that is relevant to our discussion. Suppose that the Markov chain defined by equation (6) is ψ-irreducible and aperiodic. If there exists a small set C in the state space $\mathbb{R}^d$ of $\boldsymbol{X}_n$, constants $b < \infty$, $\beta > 0$ and a function $v : \mathbb{R}^d \to [0, \infty]$ such that $v(\boldsymbol{\xi}) \geqslant 1$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$ and finite at some point of the state space satisfying the condition

$$\Delta v(\boldsymbol{\xi}) \leqslant -\beta v(\boldsymbol{\xi}) + b 1_C(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{R}^d, \tag{8}$$

then the set $S_v = \{\boldsymbol{\xi} \in \mathbb{R}^d : v(\boldsymbol{\xi}) < \infty\}$ is absorbing, that is, a Markov chain starting in S_v cannot exit this set, and full, that is, S_v supports the entire irreducible measure ψ, where

$\Delta v(\boldsymbol{\xi}) = E[v(\boldsymbol{X}_1) \mid \boldsymbol{X}_0 = \boldsymbol{\xi}] - v(\boldsymbol{\xi})$. Moreover, there exist some constants $r > 1$ and $0 < \zeta < \infty$ such that for any $\boldsymbol{\xi} \in S_v$ we have

$$\sum_{m=1}^{\infty} r^m \, \|P^{(m)}(\boldsymbol{\xi}, \cdot) - \pi(\cdot)\| \leqslant \zeta \, v(\boldsymbol{\xi}), \tag{9}$$

where π is a probability measure on $\mathcal{B}(\mathbb{R}^d)$, the invariant (stationary) probability measure for the Markov chain and

$$\|P^{(m)}(\boldsymbol{\xi}, \cdot) - \pi(\cdot)\| = \sup_{A \in \mathcal{B}(\mathbb{R}^d)} |P^{(m)}(\boldsymbol{\xi}, A) - \pi(A)|. \tag{10}$$

If equation (9) holds, the Markov chain $\boldsymbol{X}_n, n = 0, 1, \ldots,$ starting with its stationary probability π, is ergodic, and $r^m \, \|P^{(m)}(\boldsymbol{\xi}, \cdot) - \pi(\cdot)\| \to 0$ as $m \to \infty$, so that the stationary probability $\pi(A)$ can be approximated by the m-step transition probability $P^{(m)}(\boldsymbol{\xi}, A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$ and a sufficiently large m, irrespective of $\boldsymbol{\xi} \in S_v$. Also, the convergence of $P^{(m)}$ to π takes place at a uniform geometric rate, that is independent of $\boldsymbol{\xi} \in S_v$.

Ergodicity criterion for $\boldsymbol{X}_n$. The ergodic theorem in equations (8)–(10) provides a general criterion for assessing the long term behaviour of Markov chains, which can be stated as follows. *If we can find a function $v : \mathbb{R}^d \to [0, \infty]$ satisfying the condition in equation (8), then $\boldsymbol{X}_n$ is an ergodic Markov chain whose invariant measure π can be approximated by its m-step transition probability $P^{(m)}$ for a sufficiently large m (equations (9) and (10)).* The examples in the latter part of the paper are used to demonstrate the use of this criterion.

Suppose that we have applied the above criterion and showed that a Markov chain $\boldsymbol{X}_n$ is ergodic. We show now that $\boldsymbol{X}$ in equation (2) is also ergodic, so that it is sufficient to prove that $\boldsymbol{X}_n$ is an ergodic Markov chain. First, we note that the $\mathbb{R}^{d+1}$-valued sequence $\boldsymbol{V}_n = (\boldsymbol{X}_n, Z_{n+1})$ is a Markov chain, and it is straightforward to check that its transition probabilities converge to the appropriate stationary probability and, hence, that Markov chain is ergodic as well. Second, let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable bounded function and define the real-valued sequence

$$U_n = \int_{T_n}^{T_{n+1}} f(\boldsymbol{X}(t)) \, \mathrm{d}t = \int_0^{Z_{n+1}} f(\mathrm{e}^{\boldsymbol{a} z} \boldsymbol{X}_n) \, \mathrm{d}z \qquad n = 1, 2, \ldots, \tag{11}$$

where we are using a stationary, hence ergodic, version of $\{\boldsymbol{V}_n\}$. The latter expression of U_n is valid since there is no input in the time intervals between the jumps of $\boldsymbol{C}$. The sequence $\{U_n\}$ is ergodic since it is a function of the ergodic sequence $\{\boldsymbol{V}_n\}$. Hence, we have

$$P\left(\frac{1}{m}\sum_{n=0}^{m} U_n \underset{m\to\infty}{\longrightarrow} E[f(\boldsymbol{X}(t))]\right) = 1, \tag{12}$$

([2], theorem 2, p 130), that is, time and ensemble averages coincide. By the usual arguments this implies ergodicity of the process $\boldsymbol{X}$.

5. Methods for stability analysis

Two methods are considered for evaluating the long term behaviour of the solution of equation (2). The first method is based on Lyapunov exponents, and constitutes an attempt of extending a result in [6] for assessing the stability of the solution of stochastic differential equations with Gaussian white noise ([3], section 8.7) to the case of differential equations with Poisson white noise. The second method is based on the theory of stochastic stability for Markov chains (equations (8) to (10)).

5.1. Lyapunov exponent

The Itô formula for semimartingales applied to $\ln(\| \boldsymbol{X}(t) \|^2)$ gives ([3], sections 7.1 and 7.3.2)

$$\ln(\|\boldsymbol{X}(t)\|^2) - \ln(\|\boldsymbol{X}(0)\|^2) = 2\sum_{i,j=1}^{d} a_{ij} \int_0^t S_i(u)S_j(u)\,\mathrm{d}u + \sum_{0<u\leqslant t} \ln\left(1 + 2\sum_{r=1}^{d'} A_r(u-)\Delta C_r(u) + \sum_{r=1}^{d'} B_{rr}(u-)(\Delta C_r(u))^2\right), \tag{13}$$

where $\boldsymbol{S}(t) = \boldsymbol{X}(t)/\|\boldsymbol{X}(t)\|$, A_r are the coordinates of the $\mathbb{R}^d$-valued process $\boldsymbol{A}(t) = \sum_{k=1}^{d} \boldsymbol{S}(t)^T S_k(t)\boldsymbol{b}^{(k)}$ and $\boldsymbol{B}(t) = \sum_{k,l=1}^{d} (\boldsymbol{b}^{(k)})^T \boldsymbol{b}^{(l)} S_k(t)S_l(t)$ is a (d', d') matrix with entries $B_{rp}(t)$. The formula in equation (13) shows that $R(t) = \ln(\|\boldsymbol{X}(t)\|)$ is the solution of

$$R(t) - R(0) = \sum_{i,j=1}^{d} a_{ij} \int_0^t S_i(u)S_j(u)\,\mathrm{d}u + \frac{1}{2}\sum_{0<u\leqslant t} \ln\left(1 + 2\sum_{r=1}^{d'} A_r(u-)\Delta C_r(u) + \sum_{r=1}^{d'} B_{rr}(u-)(\Delta C_r(u))^2\right), \tag{14}$$

so that

$$\frac{R(t) - R(0)}{t} = \sum_{i,j=1}^{d} a_{ij} \frac{1}{t}\int_0^t S_i(u)S_j(u)\,\mathrm{d}u + \frac{1}{2t}\sum_{0<u\leqslant t} \ln\left(1 + 2\sum_{r=1}^{d'} A_r(u-)\Delta C_r(u) + \sum_{r=1}^{d'} B_{rr}(u-)(\Delta C_r(u))^2\right). \tag{15}$$

Suppose that (1) the integrals $(1/t)\int_0^t S_i(u)S_j(u)\,\mathrm{d}u$ in equation (15) converge a.s. to some constants α_{ij} as $t \to \infty$ and (2) the second summation on the right-hand side of equation (15) converges to a constant β as $t \to \infty$. Then

$$\lim_{t\to\infty} \frac{R(t) - R(0)}{t} = \sum_{i,j=1}^{d} a_{ij}\,\alpha_{ij} + \beta = \lambda_{\mathrm{L}} \quad \text{a.s.}, \tag{16}$$

so that

$$\|\boldsymbol{X}(t)\| \sim \|\boldsymbol{X}(0)\| \exp\left[t\left(\sum_{i,j=1}^{d} a_{ij}\,\alpha_{ij} + \beta\right)\right] = \|\boldsymbol{X}(0)\|\,\mathrm{e}^{\lambda_{\mathrm{L}} t} \quad \text{a.s.}, \qquad t \to \infty, \tag{17}$$

implying that $\boldsymbol{X}$ is asymptotically stable and unstable a.s. if $\lambda_{\mathrm{L}} < 0$ and $\lambda_{\mathrm{L}} > 0$, respectively.

The above approach was applied successfully in [6] to assess the stability of stationary solutions of stochastic differential equations with Gaussian white noise. In was shown that, if $\boldsymbol{X}$ is $\mathbb{R}^d$-valued diffusion processes, (1) the ratio $\ln\|\boldsymbol{X}(t)\|/t$ converges a.s as time increases indefinitely and (2) this limit may take two or more values for $d > 1$, which depend on the initial state $\boldsymbol{X}(0)$. Moreover, a method was developed in [6] for calculating the largest value of $\lim_{t\to\infty} \ln\|\boldsymbol{X}(t)\|/t$, needed to determine the long term behaviour of $\boldsymbol{X}$ and referred to as the top Lyapunov exponent.

It was not possible to extend the method in [6] to the case of general stochastic differential equations with Poisson white noise. Our attempt of extending the methodology in [6] to equations driven by Poisson white noise was unsuccessful. It was not possible to develop general criteria for validating the assumptions leading to equations (16) and (17). That these assumptions hold in some cases is demonstrated by examples later in the paper.

5.2. Stochastic stability for Markov chains

According to the geometric ergodic theorem in the previous section (equations (8) to (10)), it is sufficient to find a solution v of equation (8) for a small set C to assure that the Markov chain in equation (6) is ergodic. In this case, $P^{(m)}(\boldsymbol{\xi}, \cdot)$ in equation (9) can be used to approximate the invariant probability measure π of $\boldsymbol{X}_n$ provided that m is sufficiently large, irrespective of the starting point $\boldsymbol{X}_0 = \boldsymbol{\xi} \in S_v$. This approximation of π can be used to examine the long term behaviour of the Markov chain $\boldsymbol{X}_n$. Long term properties of $\boldsymbol{X}_n$ can also be inferred by examining samples of $\boldsymbol{X}_n$ generated by Monte Carlo simulation from the recurrence formula in equation (6).

6. Applications

Two processes defined by equation (2) with $d = 1$ and $d = 2$ are considered. Lyapunov exponents can be obtained by relatively simple calculations for the real-valued process ($d = 1$), but are difficult to find for $\mathbb{R}^d$-valued process with $d \geqslant 2$. The theory of stochastic stability for Markov chains proved adequate for (1) establishing ergodicity conditions for $\mathbb{R}^d$-valued processes defined by equation (2) and (2) assessing the long term behaviour of these processes.

6.1. An $\mathbb{R}$-valued process

Let $X(t)$, $t \geqslant 0$, be a real-valued process defined by the stochastic differential equation

$$\mathrm{d}X(t) = -\alpha X(t-)\,\mathrm{d}t + X(t-)\,\mathrm{d}C(t), \qquad t \geqslant 0, \tag{18}$$

where α is a constant, $C(t) = \sum_{k=1}^{N(t)} Y_k$ is a compound Poisson process, Y_k, $k = 1, 2, \ldots$, are independent identically distributed real-valued random variables, independent of a Poisson process N with intensity $\lambda > 0$ (equation (3)). The process X in equation (18) is referred to as the geometric compound Poisson process ([3], example 8.56).

6.1.1. Lyapunov exponent. Calculations as in the previous section using Itô's formula show that $R(t) = \ln(|X(t)|)$ satisfies the stochastic differential equation

$$\mathrm{d}R(t) = -\alpha \mathrm{d}t + \int_{\mathbb{R}} \ln(1 + y)\mathcal{M}(\mathrm{d}t, \mathrm{d}y), \tag{19}$$

where the Poisson random measure $\mathcal{M}(\mathrm{d}t, \mathrm{d}y)$ gives the number of jumps of C whose time and magnitude are in the rectangle $(t, t + \mathrm{d}t] \times (y, y + \mathrm{d}y]$ and has expectation $E[\mathcal{M}(\mathrm{d}t, \mathrm{d}y)] = \lambda \mathrm{d}t\, \mathrm{d}F_y(y)$ depending on the distribution function F_y of Y_1 ([3], section 8.7). The above equation is meaningful for $1 + Y_1 > 0$ a.s., and we assume that this inequality holds. We have (equation (17))

$$|X(t)| \sim |X(0)| \exp(-\alpha t + C^*(t)), \tag{20}$$

where $C^*(t) = \sum_{k=1}^{N(t)} \ln(1 + Y_k)$. If $E[\ln(1 + Y_1)]$ exists and is finite, then (equation (16))

$$\lambda_{\mathrm{L}} = -\alpha + \lim_{t\to\infty} \frac{C^*(t)}{t} = \alpha + \lambda E[\ln(1 + Y_1)] \text{ a.s.}, \tag{21}$$

where the final expression of λ_{L} results from the strong law of large numbers. Since for real-valued processes there is a single Lyapunov exponent, we conclude that under the condition

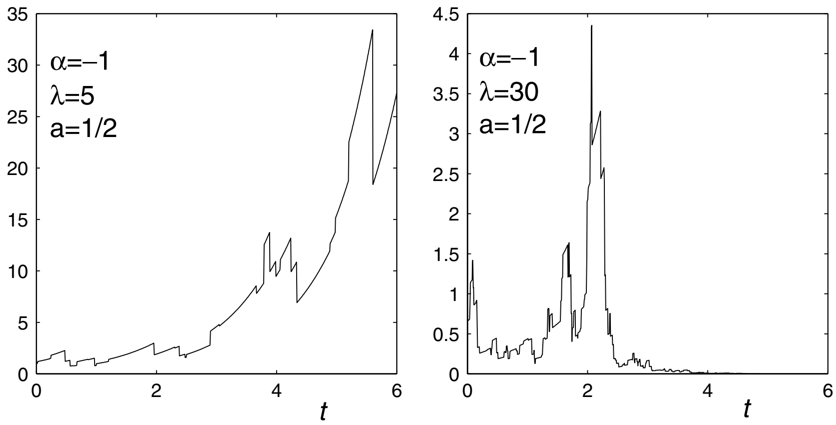

Figure 1. Samples of X in equation (18) for $\alpha = -1$, jumps in $(-1/2, 1/2)$, and two values of λ.

$1+Y_1 > 0$ a.s. the trivial solution of equation (18) is stable a.s. if $\lambda_L = -\alpha + \lambda E[\ln(1+Y_1)] < 0$, that is, the solution of this equation converges a.s. to 0 as $t \to \infty$.

Figure 1 shows two samples of X in equation (18) with $\alpha = 1$, jumps Y_k uniformly distributed in $(-1/2, 1/2)$, intensities $\lambda = 5$ (left panel) and $\lambda = 30$ (right panel), and initial state $X(0) = 1$. The Lyapunov exponents are $\lambda_L = 0.7725$ and $\lambda_L = -0.3650$ for $\lambda = 5$ and $\lambda = 30$, respectively, since $E[\ln(1+Y_1)] \simeq -0.0455$. The long term behaviour of the samples of X illustrated in figure 1 is consistent with our prediction based on the Lyapunov exponent λ_L.

6.1.2. Stochastic stability for Markov chains. Let $T_0 = 0$ and denote by $0 \leqslant T_1 < T_2 < \cdots$ the jump times of the Poisson process N. The Markov chain X_n, $n = 0, 1, \ldots,$ in equation (6) is defined by the recurrence formula

$$X_n = (1 + Y_n) X_{n-1}\, \mathrm{e}^{-\alpha Z_n}, \qquad n = 1, 2, \ldots, \tag{22}$$

where X_0 denotes the initial state and $Z_n = T_n - T_{n-1}, n = 1, 2, \ldots,$ are independent exponential random variables with mean $1/\lambda$.

The recurrence formula in equation (22) gives

$$|X_n| \mid (X_0 = \xi) = |\xi| \prod_{k=1}^{n} |1 + Y_k|\, \mathrm{e}^{-\alpha \sum_{k=1}^{n} Z_k} = |\xi| \exp\left(\sum_{k=1}^{n} [\ln|1+Y_k| - \alpha Z_k]\right). \tag{23}$$

Let $R_n = \sum_{k=1}^{n} [\ln|1+Y_k| - \alpha Z_k]$, $n = 1, 2, \ldots,$ be a random walk starting at $R_0 = 0$. If the expectation

$$E[\ln|1+Y_1| - \alpha Z_1] = E[\ln|1+Y_1|] - \alpha/\lambda = \lambda_L/\lambda$$

is positive, negative and 0, then $R_n \to +\infty$ a.s. as $n \to \infty$, $R_n \to -\infty$ a.s. as $n \to \infty$, and $\lim_{n\to\infty} R_n$ does not exist, respectively ([3], section 2.14). Therefore, if $\xi \neq 0$, $|X_n| \to \infty$ a.s. as $n \to 0$ and $X_n \to 0$ a.s. as $n \to \infty$ for $E[\ln|1+Y_1|] - \alpha/\lambda > 0$ and $E[\ln|1+Y_1|] - \alpha/\lambda < 0$, respectively, in agreement with the long term behaviour of the process X in equation (18) predicted by the Lyapunov exponent in equation (21).

The following three examples show that one can use the ergodic theorem discussed earlier in the paper to demonstrate stability of the Markov chain in equation (22) under appropriate conditions.

Example 1. Suppose that $\alpha > 0$, $E[Y_1] = 0$ and $1 + Y_1 > 0$ a.s., so that X_n in equation (22) takes values in $[0, \infty)$ and $(-\infty, 0]$ for $X_0 > 0$ and $X_0 < 0$, respectively. Hence, we can assume $X_0 = \xi > 0$ without loss of generality.

To apply the ergodic theorem we need to select a function v, for example, $v(\xi) = \xi + 1$, and a small set C for which the condition in equation (8) is satisfied. Simple considerations show that $C = [0, 1]$ is not a small set for X_n since it includes $\{0\}$. If $\{0\}$ is eliminated from C, for example, we replace C with $C^* = [\varepsilon, 1]$, $0 < \varepsilon < 1$, then the condition in equation (8) gives $\Delta v(\xi) + \beta v(\xi) = \beta \leqslant 0$ for $\xi = 0 \notin C^*$, so that there is no $\beta > 0$ satisfying this equation. We apply the ergodic theorem to a modified version of X_n.

Let $\tilde{X}_n$, $n = 0, 1, \ldots$, with initial state $\tilde{X}_0 = \xi$ be defined by the recurrence formula

$$\tilde{X}_n = (1 + Y_n)\tilde{X}_{n-1}\,\mathrm{e}^{-\alpha Z_n} + B_n, \qquad n = 1, 2, \ldots, \tag{24}$$

where B_n are independent copies of an exponential random variables with expectation $1/\mu$, $\mu > 0$, which are independent of the random variables $\{Y_n\}$ and $\{Z_n\}$. Under the assumptions $1 + Y_1 > 0$ a.s. and $\tilde{X}_0 = \xi > 0$, we have $0 \leqslant X_n \leqslant \tilde{X}_n$ at all times, so that if $\tilde{X}_n$ is stable, so is X_n. We now show that the condition in equation (8) with $v(\xi) = \xi + 1$ and $C = [0, 1]$ holds for $\tilde{X}_n$, and that C is a small set for $\tilde{X}_n$. We then conclude that $\tilde{X}_n$ is an ergodic Markov chain whose invariant measure π can be approximated by its m-step transition probability $P^{(m)}$ for a sufficiently large m (equations (9) and (10)). The stability conclusion extends to the original Markov chain X_n by the above considerations.

Let $v(\xi) = \xi + 1$, $\tilde{X}_0 = \xi \geqslant 0$ and $C = [0, 1]$. Assuming $\alpha + \lambda > 0$, we have

$$\Delta v(\xi) = E[v(\tilde{X}_1) \mid \tilde{X}_0 = \xi] - v(\xi) = -\frac{\xi\alpha}{\lambda + \alpha} + \frac{1}{\mu}. \tag{25}$$

The condition, $\Delta v(\xi) + \beta v(\xi) \leqslant 0$, $\xi \geqslant 1$, in equation (8) requires

$$\beta \leqslant \frac{\xi}{\xi + 1}\frac{\alpha}{\lambda + \alpha} - \frac{1}{\xi + 1}\frac{1}{\mu},$$

for $\xi \notin C$, so that we can take

$$\beta = \frac{1}{2}\left(\frac{\alpha}{\lambda + \alpha} - \frac{1}{\mu}\right) \tag{26}$$

under the condition $\mu > 1 + \lambda/\alpha$ implying $\beta > 0$. The condition in equation (8) also requires $\Delta v(\xi) + \beta v(\xi) - b \leqslant 0$ for $\xi \in C$, β in equation (26), and some $b < \infty$. This requirement can be satisfied for

$$b = \frac{1}{2}\left(\frac{\alpha}{\lambda + \alpha} + \frac{1}{\mu}\right). \tag{27}$$

It remains to show that $C = [0, 1]$ is a small set for $\tilde{X}_n$. We have $\tilde{X}_1 \mid (\tilde{X}_0 = \xi) = A_1 + B_1$ with the notation $A_1 = (1 + Y_1)\xi\,\mathrm{e}^{-\alpha Z_1}$. Since the random variables A_1 and B_1 are positive and independent of each other, we have $P(\tilde{X}_1 \leqslant x \mid \tilde{X}_0 = \xi) = \int_0^x F_{B_1}(x - u) f_{A_1}(u)\,\mathrm{d}u$, so that

$$p(\xi, x) = \frac{\mathrm{d}}{\mathrm{d}x} P(\tilde{X}_1 \leqslant x \mid \tilde{X}_0 = \xi) = \int_0^x f_{B_1}(x - u) f_{A_1}(u)\,\mathrm{d}u \geqslant \mu\,\mathrm{e}^{-\mu x} F_{A_1}(x),$$

where f_U and F_U denote the density and the distribution of a random variable U, respectively. Since

$$F_{A_1}(x) = P(Y_1 \leqslant x\,\mathrm{e}^{\alpha Z_1}/\xi - 1) \geqslant \int_0^\infty F_{Y_1}(x\,\mathrm{e}^{\alpha z} - 1) f_{Z_1}(z)\,\mathrm{d}z,$$

we have

$$p(\xi, x) \geqslant \mu\, e^{-\mu x} \int_0^\infty F_{Y_1}(x\, e^{\alpha Z_1} - 1) f_{Z_1}(z)\, dz > 0, \qquad \xi \in C = [0, 1],$$

so that C is a small set for $\tilde{X}_n$.

We note that the state X in equation (18) with $\alpha = -1$, $Y_1 \sim U(-1/2, 1/2)$ and $\lambda = 30$ is stable since the Lyapunov λ_L in equation (21) is negative. A sample of X with these parameters is shown in the right panel of figure 1. The value of β in equation (26) corresponding to these parameters is negative. However, we cannot conclude that the Markov chain $\tilde{X}_n$ is not stable since the ergodic theorem provides only sufficient conditions for stability. Other selection of the pair (v, C) may and do show that the chain is stable.

Example 2. Now $1 + Y_1$ can take positive and negative values, so that the support of X_n is the entire real line. We will no longer assume that $\alpha > 0$ but, instead, only that $\alpha + \lambda > 0$. Furthermore, we no longer assume that $EY_1 = 0$. Instead we assume that $E|1+Y_1| < 1+\alpha/\lambda$. As mentioned in the previous example, it is not possible to apply the condition in equation (8) to the Markov chain X_n. We apply this condition to a Markov chain $\tilde{X}_n$, $n = 0, 1, \ldots,$ defined by

$$\tilde{X}_n = (1 + Y_n)\tilde{X}_{n-1}\, e^{-\alpha Z_n} + B_n, \qquad n = 1, 2, \ldots, \tag{28}$$

where $\tilde{X}_0 = X_0$ and B_n are independent copies of a Gaussian variable with mean 0 and variance σ_B^2, that are independent of the random variables $\{Y_n\}$ and $\{Z_n\}$. We show that, if σ_B^2 is small enough, $\tilde{X}_n$ is stable and that the difference between $\tilde{X}_n$ and X_n is stable as well. Hence, X_n is stable.

We first apply the condition of equation (8) with $v(\xi) = |\xi| + 1$ and $C = [-1, 1]$ to the Markov chain $\tilde{X}_n$. We have

$$\begin{aligned} \Delta v(\xi) &= E[|\tilde{X}_1| + 1 \mid \tilde{X}_0 = \xi] - (|\xi| + 1) = E[|(1 + Y_1)\xi\, e^{-\alpha Z_1} + B_1|] - |\xi| \\ &\leqslant E[|1 + Y_1||\xi|\, e^{-\alpha Z_1} + |B_1|] - |\xi| = |\xi| \left(\frac{\lambda E[|1 + Y_1|]}{\lambda + \alpha} - 1 \right) + E[|B_1|]. \end{aligned} \tag{29}$$

To satisfy the condition in equation (8) with $C = [-1, 1]$ and $\xi \notin C$ we choose $\beta > 0$ such that

$$\Delta v(\xi) + \beta v(\xi) \leqslant |\xi| \left(\frac{\lambda E[|1 + Y_1|]}{\lambda + \alpha} - 1 \right) + E[|B_1|] + \beta(|\xi| + 1) \leqslant 0$$

or

$$\beta \leqslant \frac{|\xi|}{|\xi| + 1} \left(1 - \frac{\lambda E[|1 + Y_1|]}{\lambda + \alpha} \right) - \frac{E[|B_1|]}{|\xi| + 1}$$

for all $\xi \notin C$. Therefore, we take

$$\beta = \frac{1}{2} \left[\left(1 - \frac{\lambda E[|1 + Y_1|]}{\lambda + \alpha} \right) - E[|B_1|] \right]. \tag{30}$$

The expression of β in equation (30) is strictly positive if $1 - E[|1 + Y_1|]/(1 + \alpha/\lambda) - \sqrt{2/\pi}\,\sigma_B > 0$. This condition is satisfied if, for example, $\alpha = 2$, $\lambda = 1$, Y_1 is uniformly distributed in $(-1, 1)$ and $\sigma_B = 0.5$, in which case $\beta = 0.2667$.

To satisfy the condition in equation (8) for $\xi \in C$ we choose b such that

$$\Delta v(\xi) + \beta v(\xi) - b \leqslant |\xi| \left(\frac{\lambda E[|1 + Y_1|]}{\lambda + \alpha} - 1 \right) + E[|B_1|] + \beta(|\xi| + 1) - b \leqslant 0$$

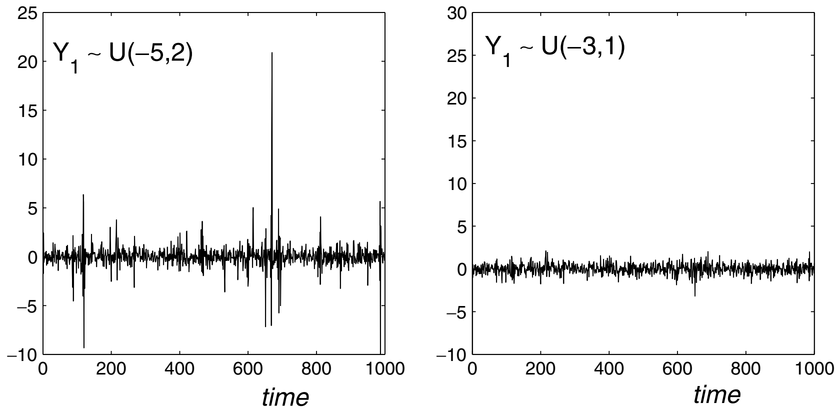

Figure 2. Samples of Δ_n for $\alpha = 20, \lambda = 10, \sigma_B = 0.5, Y_1 \sim U(-5, 2)$ (left panel) and $Y_1 \sim U(-3, 1)$ (right panel).

or

$$b \geqslant \frac{1-|\xi|}{2}\left[\left(1-\frac{\lambda E[|1+Y_1|]}{\lambda+\alpha}\right)+E[|B_1|]\right],$$

for all $\xi \in C = [-1, 1]$, so that we can take

$$b = \frac{1}{2}\left[\left(1-\frac{\lambda E[|1+Y_1|]}{\lambda+\alpha}\right)+E[|B_1|]\right]. \tag{31}$$

Now we show that $C = [-1, 1]$ is a small set for $\tilde{X}_n$. Observe that the transition density is given here by

$$p(\xi, x) = \frac{1}{\sigma_B\sqrt{2\pi}} E\left[\exp\left(-\frac{(x-(1+Y_1)\xi\,\mathrm{e}^{-\alpha Z_1})^2}{2\sigma_B^2}\right)\right].$$

Choose $\rho > 0$ such that $q = P(|1 + Y_1| \leqslant \rho) > 0$. Then for $x > 2\rho \max(1, \mathrm{e}^{-\alpha})$ we have

$$p(\xi, x) \geqslant \frac{1}{\sigma_B\sqrt{2\pi}} q(1-\mathrm{e}^{-\lambda})\,\mathrm{e}^{-x^2/8\sigma_B^2},$$

a uniform in ξ lower bound that shows that C is a small set for $\tilde{X}_n$.

Therefore, $\tilde{X}_n$ is stable. Observe now that the difference $\Delta_n = \tilde{X}_n - X_n$ satisfies the recurrence equation

$$\Delta_n = (1+Y_n)\Delta_{n-1}\,\mathrm{e}^{-\alpha Z_n} + B_n, \qquad n = 1, 2, \ldots, \tag{32}$$

which is exactly the same equation as equation (28), but with a different initial state $\Delta_0 = 0$. Therefore, the above argument shows also that the difference between $\tilde{X}_n$ and X_n is stable, and so X_n is stable as well.

Figure 2 shows two samples of Δ_n in equation (32) for $\alpha = 20, \lambda = 10, \sigma_B = 0.5, Y_1 \sim U(-5, 2)$ (left panel) and $Y_1 \sim U(-3, 1)$ (right panel). The notable differences between these two samples are caused by the size of the jumps Y_1 in the definition of the driving noise.

Example 3. Here we show how the ergodic theorem for Markov chains works in the most general case, that of

$$E[\ln|1+Y_1|] < \alpha/\lambda. \tag{33}$$

We will assume that $E|1+Y_1|^\theta < \infty$ for some $\theta > 0$. Even though the function v used in the previous example may not always work here, we will see that the function $v(\xi) = |\xi|^p + 1$ will work for some $p > 0$. We will, once again, apply the Markov chain theory to the modified Markov chain $\tilde{X}_n, n = 0, 1, \ldots$ given in equation (28). We still use $C = [-1, 1]$, which, as we know from the previous example, is a small set for $\tilde{X}_n$.

Note that for $p \in (0, 1]$ we have

$$\begin{aligned}\Delta v(\xi) &= E[|\tilde{X}_1|^p + 1 \mid \tilde{X}_0 = \xi] - (|\xi|^p + 1) = E[|(1+Y_1)\xi\,\mathrm{e}^{-\alpha Z_1} + B_1|^p] - |\xi|^p \\ &\leqslant |\xi|^p[E(|1+Y_1|\,\mathrm{e}^{-\alpha Z_1})^p - 1] + E[|B_1|^p]. \end{aligned} \tag{34}$$

Since $(U^p - 1)/p \leqslant (U^\theta - 1)/\theta$, $p \in (0, \theta]$, for $U \geqslant 1$, and $(1 - U^p)/p \leqslant -\log U$ for $0 < U < 1$, we have

$$E\left[\frac{U^p - 1}{p}\right] \to E[\log U]$$

as $p \downarrow 0$ by dominated convergence. Therefore, under the assumption in equation (33) there is $p \in (0, 1]$ such that

$$E[|1+Y_1|\,\mathrm{e}^{-\alpha Z_1}]^p - 1 := -c < 0.$$

Fixing that p, we have for all $\xi \notin C$

$$\Delta v(\xi) + \beta v(\xi) \leqslant -c|\xi|^p + E[|B_1|^p] + \beta(|\xi|^p + 1) < 0$$

if we choose $\beta < c/2$ and the variance σ_B^2 of B_1 small enough. Similarly, it is easy to choose b that will satisfy the condition in equation (8) with $\xi \in C$. Therefore, the Markov chain $\tilde{X}_n$ is stable. Since the difference $\Delta_n = \tilde{X}_n - X_n$ satisfies the same recurrence equation as $\tilde{X}_n$ does, Δ_n is stable as well and, hence, so is $X_n = \tilde{X}_n + \Delta_n$.

We have established conditions under which the state X of the dynamic system in equation (18) is stable based on methods using Lyapunov exponents and an ergodic theorem for Markov chains. It was found using Lyapunov exponents that X is stable if $-\alpha + \lambda E[\ln|1+Y_1|] < 0$. The conditions for the stability of the associated Markov chain X_n in equation (22) depend on the selection of the function v and of the small set C used in equation (8). An inadequate choice of (v, C) may result in very restrictive stability conditions for X_n.

6.2. *An $\mathbb{R}^2$-valued process*

Let $\boldsymbol{X}(t)$, $t \geqslant 0$, be defined by

$$\begin{cases}\mathrm{d}X_1(t) = a_1X_1(t-)\,\mathrm{d}t + X_1(t-)\,\mathrm{d}C_1(t) + X_2(t-)\,\mathrm{d}C_2(t) \\ \mathrm{d}X_2(t) = a_2X_2(t-)\,\mathrm{d}t + X_2(t-)\,\mathrm{d}C_1(t) - X_1(t-)\,\mathrm{d}C_2(t),\end{cases} \tag{35}$$

where a_1, a_2 are some constants, $C_r, r = 1, 2$, are two-independent compound Poisson processes defined by equation (3), N_r denote two mutually independent Poisson processes with intensity λ_r and $Y_{r,q}, q = 1, 2, \ldots,$ are independent identically distributed random variables for $r = 1, 2$, which do not depend on the processes N_r. An alternative form of the defining equation for $\boldsymbol{X} = (X_1, X_2)$ is

$$\mathrm{d}\begin{bmatrix}X_1(t)\\X_2(t)\end{bmatrix} = \begin{bmatrix}a_1X_1(t-)\\a_2X_2(t-)\end{bmatrix}\mathrm{d}t + \begin{bmatrix}X_1(t-) & X_2(t-)\\X_2(t-) & -X_1(t-)\end{bmatrix}\mathrm{d}\boldsymbol{C}(t), \tag{36}$$

where $\boldsymbol{C}(t) = \sum_{k=1}^{N(t)}\boldsymbol{V}_k$, N denotes a Poisson process with intensity $\lambda = \lambda_1 + \lambda_2$ and $\boldsymbol{V}_k$ are $\mathbb{R}^2$-valued-independent random variables equal in distribution to $(Y_{1,1}, 0)$ and $(0, Y_{2,1})$ with the probabilities λ_1/λ and λ_2/λ, respectively (equation (4)).

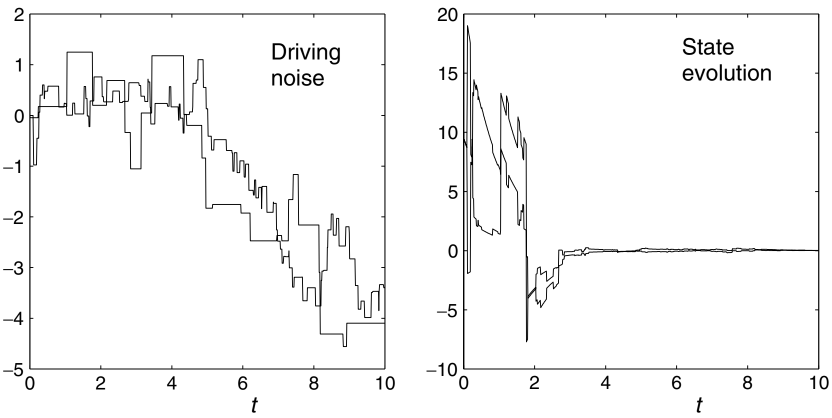

Figure 3. Samples of $\boldsymbol{C}$ and $\boldsymbol{X}$ for $X_1(0) = X_2(0) = 10$, $a_1 = a_2 = -1$, $\lambda_1 = 1$ and $\lambda_2 = 2$.

6.2.1. Lyapunov exponent. The Itô formula for semimartingales applied to $\ln(\|\boldsymbol{X}(t)\|^2)$ gives

$$\ln(\|\boldsymbol{X}(t)\|^2) - \ln(\|\boldsymbol{X}(0)\|^2) = 2a_1 \int_0^t \frac{X_1(u)^2}{\|\boldsymbol{X}(u)\|^2}\,\mathrm{d}u + 2a_2 \int_0^t \frac{X_2(u)^2}{\|\boldsymbol{X}(u)\|^2}\,\mathrm{d}u + \bar{C}(t) \tag{37}$$

following simple but lengthy calculations, where $\bar{C}(t) = \sum_{k=1}^{N(t)} \bar{Y}_k$ and $\bar{Y}_k$ are independent copies of a random variable $\bar{Y}_1$ equal to $\ln\left(1 + Y_{1,1}^2\right)$ and $\ln\left(1 + Y_{2,1}^2\right)$ with probabilities λ_1/λ and λ_2/λ, respectively. The above equation implies that $R(t) = \ln(\|\boldsymbol{X}(t)\|)$ is given by

$$R(t) - R(0) = \int_0^t (a_1 S_1(u)^2 + a_2 S_2(u)^2)\,\mathrm{d}u + \frac{1}{2}\bar{C}(t), \tag{38}$$

so that

$$\frac{R(t) - R(0)}{t} = \frac{1}{t}\int_0^t (a_1 S_1(u)^2 + a_2 S_2(u)^2)\,\mathrm{d}u + \frac{1}{2t}\bar{C}(t), \tag{39}$$

where $\boldsymbol{S}(t) = \boldsymbol{X}(t)/\ \|\boldsymbol{X}(t)\|$.

If $\bar{Y}_1$ is integrable, then $\lim_{t\to\infty}(1/(2t))\bar{C}(t) = (\lambda/2)E[\bar{Y}_1]$ almost surely. If $\boldsymbol{X}$ in equation (36) is ergodic, then the integral on the right-hand side of the above equation converges a.s. to a constant $\bar{\alpha}$ as $t \to \infty$ since it has the form $(1/t)\int_0^t f(\boldsymbol{X}(u))\,\mathrm{d}u$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is a bounded Borel measurable function. Under these conditions we have

$$\lim_{t\to\infty} \frac{R(t) - R(0)}{t} = \bar{\alpha} + \frac{\lambda}{2}E[\bar{Y}_1] = \lambda_{\mathrm{L}}\text{a.s.}, \tag{40}$$

so that $\|\boldsymbol{X}(t)\| \sim \|\boldsymbol{X}(0)\|\ \exp[(\bar{\alpha} + (\lambda/2)E[\bar{Y}_1])t]$ a.s. for large times t implying that the trivial stationary solution is stable a.s. if $\bar{\alpha} + (\lambda/2)E[\bar{Y}_1] < 0$.

Figures 3 and 4 show samples of the driving noise $\boldsymbol{C}$ with $\lambda_1 = 1$, $\lambda_2 = 2$, and jumps $Y_{1,r}$ uniformly distributed in $(-\sqrt{3/\lambda_r}, \sqrt{3/\lambda_r})$, $r = 1, 2$, and the corresponding samples of $\boldsymbol{X}$ for $\boldsymbol{X}(0) = (10, 10)$ and system coefficients $a_1 = a_2 = -1$ and $a_1 = -a_2 = 1$. The estimates of the Lyapunov exponents for these samples of $\boldsymbol{X}$, are $\lambda_{\mathrm{L}} = -0.4103$ for $a_1 = a_2 = -1$ and $\lambda_{\mathrm{L}} = 0.9763$ for $a_1 = -a_2 = -1$, are consistent with the evolution in time of the samples of $\boldsymbol{X}$. The norm $\|\boldsymbol{X}(t)\|$ of the state $\boldsymbol{X}(t)$ approaches 0 as t increases for $a_1 = a_2 = -1$ and increases in time for $a_1 = -a_2 = 1$. However, we cannot conclude that the trivial solution equation (35) with coefficients $a_1 = a_2 = -1$ and the above input noise is stable a.s. since Lyapunov coefficients larger than $\lambda_{\mathrm{L}} = -0.4103$ may be found for other initial states $\boldsymbol{X}(0)$.

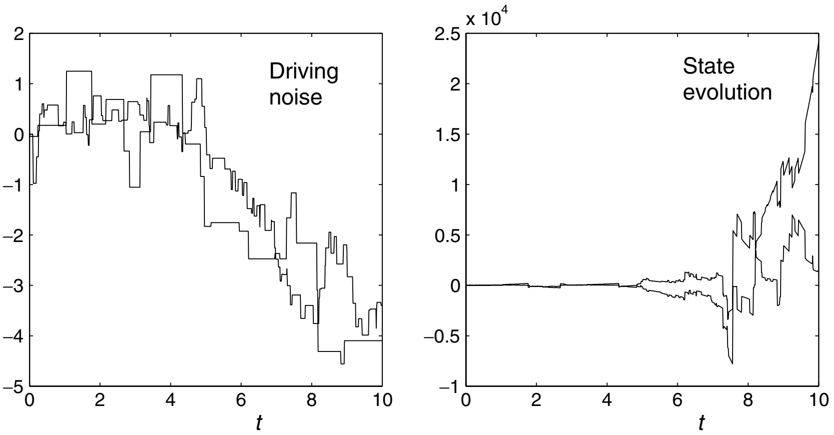

Figure 4. Samples of $\boldsymbol{C}$ and $\boldsymbol{X}$ for $X_1(0) = X_2(0) = 10$, $a_1 = -a_2 = 1$, $\lambda_1 = 1$ and $\lambda_2 = 2$.

6.2.2. Stochastic stability for Markov chains. Let $T_0 = 0$ and $0 \leqslant T_1 < T_2 < \cdots$ denote the jump times of N and let $X_{i,n} = X_i(T_n)$, $i = 1, 2$. The sequence $Z_n = T_n - T_{n-1}$, $n = 1, 2, \ldots,$ of times between consecutive jumps of N is independent and exponentially distributed with mean $1/\lambda$. Since $X_i(T_n-) = X_{i,n-1}\exp(a_i Z_n)$, $i = 1, 2$, we have the recurrence formula

$$\begin{bmatrix} X_{1,n} \\ X_{2,n} \end{bmatrix} = \begin{bmatrix} X_{1,n-1}\,\mathrm{e}^{a_1 Z_n} \\ X_{2,n-1}\,\mathrm{e}^{a_2 Z_n} \end{bmatrix} + \begin{bmatrix} X_{1,n-1}\,\mathrm{e}^{a_1 Z_n} & X_{2,n-1}\,\mathrm{e}^{a_2 Z_n} \\ X_{2,n-1}\,\mathrm{e}^{a_2 Z_n} & -X_{1,n-1}\,\mathrm{e}^{a_1 Z_n} \end{bmatrix} \boldsymbol{V}_n \tag{41}$$

for the Markov chain $\boldsymbol{X}_n = (X_{1,n}, X_{2,n})$. This formula gives

$$\boldsymbol{X}_n = \begin{bmatrix} 1 + V_{1,n} & V_{2,n} \\ -V_{2,n} & 1 + V_{1,n} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{a_1 Z_n} & 0 \\ 0 & \mathrm{e}^{a_2 Z_n} \end{bmatrix} \boldsymbol{X}_{n-1} = \boldsymbol{W}_n \boldsymbol{Z}_n \boldsymbol{X}_{n-1} = \boldsymbol{U}_n \boldsymbol{X}_{n-1}, \tag{42}$$

where $\boldsymbol{W}_n$ and $\boldsymbol{Z}_n$ denote the above $(2, 2)$ matrices depending on the random variables $\boldsymbol{V}_n$ and Z_n, respectively, and $\boldsymbol{U}_n = \boldsymbol{W}_n \boldsymbol{Z}_n$.

As in the previous three examples it is not possible to apply directly the condition in equation (8) to the Markov chain $\boldsymbol{X}_n$ to assess its stability. We consider a modified Markov chain $\tilde{\boldsymbol{X}}_n$, $n = 0, 1, \ldots,$ defined by

$$\tilde{\boldsymbol{X}}_n = \boldsymbol{W}_n \boldsymbol{Z}_n \tilde{\boldsymbol{X}}_{n-1} + \boldsymbol{B}_n = \boldsymbol{U}_n \tilde{\boldsymbol{X}}_{n-1} + \boldsymbol{B}_n, \qquad n = 1, 2, \ldots, \tag{43}$$

with initial state $\tilde{\boldsymbol{X}}_0 = \boldsymbol{\xi}$, where $\boldsymbol{B}_n$ are independent copies of an $\mathbb{R}^2$-valued random variable $\boldsymbol{B}$ with $E[\boldsymbol{B}] = \mathbf{0}$ that are independent of the random variables $\{\boldsymbol{V}_n\}$ and $\{Z_n\}$. As above, if we show that $\tilde{\boldsymbol{X}}_n$ satisfies the conditions of the ergodic theorem discussed earlier in the paper, then so does $\boldsymbol{\Delta}_n = \tilde{\boldsymbol{X}}_n - \boldsymbol{X}_n$, and then the Markov chain $\boldsymbol{X}_n$ is stable.

We now apply the condition in equation (8) with $v(\boldsymbol{\xi}) = |\xi_1|^p + |\xi_2|^p + 1$ for some $0 < p \leqslant 1$ and $C = [-1, 1] \times [-1, 1]$ to the Markov chain $\tilde{\boldsymbol{X}}_n$. We have

$$\begin{aligned} \Delta v(\boldsymbol{\xi}) &= E[|\tilde{X}_{1,1}|^p + |\tilde{X}_{1,2}|^p + 1 \mid \tilde{\boldsymbol{X}}_0 = \boldsymbol{\xi}] - (|\xi_1|^p + |\xi_2|^p + 1) \\ &\leqslant E[|U_{1,11}|^p]|\xi_1|^p + E[|U_{1,12}|^p]|\xi_2|^p + E[|U_{1,21}|^p]|\xi_1|^p \\ &\quad + E[|U_{1,22}|^p]|\xi_2|^p + E[|B_{1,1}|^p] + E[|B_{1,2}|^p] - |\xi_1|^p - |\xi_2|^p \\ &\leqslant (|\xi_1|^p + |\xi_2|^p) \left\{ \frac{\lambda}{\max(0, \lambda - pa)} E[|1 + V_{1,1}|^p + |V_{1,2}|^p] - 1 \right\} \\ &\quad + E[|B_{1,1}|^p] + E[|B_{1,2}|^p], \end{aligned} \tag{44}$$

where $U_{1,ij}$, $i, j = 1, 2$, denote the entries of $\boldsymbol{U}_1$ and $a = \max(a_1, a_2)$. The above inequality follows from the expression of the coordinates of $\tilde{\boldsymbol{X}}_1 \mid (\tilde{\boldsymbol{X}}_0 = \boldsymbol{\xi}) = \boldsymbol{U}_1\boldsymbol{\xi} + \boldsymbol{B}_1$, for example, the first coordinate of this vector is $U_{1,11}\xi_1 + U_{1,12}\xi_2 + B_{1,1}$.

Suppose that there is $0 < p \leqslant 1$ such that

$$\frac{\lambda}{\max(0, \lambda - pa)} E[|1 + V_{1,1}|^p + |V_{1,2}|^p] - 1 = -c < 0. \tag{45}$$

Then the condition in equation (8) with the above function v and $\boldsymbol{\xi} \notin C = [-1, 1] \times [-1, 1]$ will be valid for any $0 < \beta < c$ as long as we select, say, the entries of $\boldsymbol{B}_1$ to be independent Gaussian variables with mean 0 and sufficiently small variance σ_B^2. Furthermore, as above we see that taking, say,

$$b = 4\beta \tag{46}$$

will fulfil the requirement $\Delta v(\boldsymbol{\xi}) + \beta v(\boldsymbol{\xi}) - b \leqslant 0$ for $\boldsymbol{\xi} \in C$.

One situation where the condition in equation (45) is satisfied is that where

$$E[\ln|1 + V_{1,1}|] + \frac{\max(a_1, a_2)}{\lambda} < 0 \tag{47}$$

and the 'disturbance' provided by $V_{1,2}$ is small enough, for example, $V_{1,2}$ is normal with a small variance, as we can see using the same argument as in example 3 above.

It remains to show that $C = [-1, 1] \times [-1, 1]$ is a small set for $\tilde{\boldsymbol{X}}_n$. The recurrence formula in equation (43) and the assumption that the entries of $\boldsymbol{B}_1$ are independent Gaussian variables with mean 0 and variance σ_B^2 give the following expression for the transition density:

$$p((\xi_1, \xi_2), (x_1, x_2)) = \frac{1}{2\pi\sigma_B^2} E\left[\exp\left(-\frac{1}{2\sigma_B^2}((x_1 - (1 + V_{1,1})\xi_1 e^{a_1 Z} - V_{1,2}\xi_2 e^{a_2 Z})^2 + (x_2 - (1 + V_{1,1})\xi_2 e^{a_2 Z} - V_{1,2}\xi_1 e^{a_1 Z})^2)\right)\right].$$

Choose $\rho > 0$ such that $q = \min(P(|1 + V_{1,1}| \leqslant \rho), P(|V_{1,2}| \leqslant \rho) > 0$. Then for $x_i > 3\rho \max(1, e^{a_1}, e^{a_2})$, $i = 1, 2$, we have

$$p((\xi_1, \xi_2), (x_1, x_2)) \geqslant \frac{1}{2\pi\sigma_B^2} e^{-(x_1^2 + x_2^2)/18\sigma_B^2},$$

a uniform in (ξ_1, ξ_2) lower bound that shows that C is a small set for $\tilde{\boldsymbol{X}}_n$.

7. Conclusions

Two methods have been considered for assessing the stability of the trivial solution of linear stochastic differential equations driven by Poisson white noise, interpreted as the formal derivative of a compound Poisson process. The first method attempts to extend the method in [6] developed for diffusion processes defined by linear stochastic differential equations, and is based on Itô's formula for semimartingales and Lyapunov exponents. The method is adequate for real-valued processes, but encounters difficulties when applied to $\mathbb{R}^d$-valued processes for $d > 1$, since there may be two or more Lyapunov exponents for these processes, the top Lyapunov exponent is needed for stability analysis, and it was not possible to develop an algorithm for calculating this Lyapunov exponent.

The second method uses a geometric ergodic theorem for Markov chains to assess the long term behaviour of a Markov chain $\boldsymbol{X}_n$, $n = 0, 1, \ldots$, associated with the solution $\boldsymbol{X}(t)$, $t \geqslant 0$, of a linear stochastic differential equation driven by Poisson white noise. It was shown that if $\boldsymbol{X}_n$ is ergodic so is $\boldsymbol{X}(t)$. Conditions have been established to assess whether $\boldsymbol{X}_n$ is an ergodic

Markov chain and characterize the long term behaviour of $\boldsymbol{X}_n$. Two numerical examples have been presented to illustrate the application of the two methods. The examples demonstrate limitations of the first method and show that the second method using the associated Markov chain $\boldsymbol{X}_n$ is general.

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